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Cascading dynamics in double-layer hypergraphs with higher-order inter-layer interdependencies

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ABSTRACT

Higher-order interactions are ubiquitous in complex systems where groups of elements interact collectively rather than through simple pairwise connections. This study investigates how higher-order inter-layer interdependencies affect the robustness and cascading dynamics in double-layer hypergraphs, in which groups of nodes in one layer are interdependent with groups of nodes in the other layer. We analyze how the average size of these interdependency groups across layers influences system robustness and the nature of phase transitions. First, as the average size of higher-order inter-layer interdependency groups increases, the system undergoes a change from a second-order to a first-order phase transition. Second, in scale-free hypergraphs, larger interdependency groups can cause two consecutive first-order transitions, indicating nontrivial dynamical behavior that may affect the predictability and controllability of the system. Finally, the impact of higher-order inter-layer interdependencies on system robustness is not straightforward, i.e., it does not monotonically increase or decrease across both scale-free and random hypergraphs. Specifically, intermediate-sized interdependency groups result in the most vulnerable system, as indicated by the highest critical point for system disintegration. These results give useful views into how multilayer network architectures with higher-order interdependencies affect the resilience and stability of interconnected systems, offering practical guidance for designing more robust infrastructure networks.

1. Introduction

In recent years, there has been a heightened concentration on the study of multilayer networks to understand the robustness of complex systems in terms of cascading failures [1]. Multilayer networks consist of multiple layered functional systems with cross-layer interdependencies, which provide a more appropriate description of many real-world systems than simple single-layer networks [2,3]. These networks are present across diverse domains of complex systems, including critical infrastructures such as power grids and communication networks [4–7] as well as social [8], transportation [9,10], biological [11], and financial systems [12]. In such systems, the failure of nodes in one layer frequently leads to the malfunction of elements in other layers because of their interdependencies, which in turn triggers cascading failures [13]. Understanding the robustness of multilayer networks has become a hot area of research, holding key implications for averting large-scale systemic failures [14,15].

In complex systems, interactions among nodes contain not only pairwise interactions but also higher-order interactions that involving multiple nodes [16]. Generally, these higher-order interactions cannot be decomposed into a linear combination of pairwise interactions, which allows them to describe much more complex interactions in the real world [17,18]. Simplicial complex is one of the effective frameworks for describing higher-order interactions, which naturally extend pairwise connections to higher-dimensional simplices. Recent researches have explored cascading dynamics within single-layer simplicial complexes [19], interdependent [20] and multilayer simplicial complexes [21] based on percolation theory. These studies assume that nodes within the same simplex exhibit complementary relationships, i.e., the failure of one node can trigger the malfunction of all other nodes in the same simplex. This modelling approach captures some essential features of higher-order interactions and clarifies the fragility of such systems when a cascading failure occurs. Hypergraphs are another useful tool for modelling higher-order interactions, which also allow for the representation of higher-order interactions involving multiple nodes [16,22,23]. Using hypergraph model and percolation theory, researchers have investigated the dynamics of cascading failures under different mechanisms of microscopic interactions, such as node-to-node [24], node-to-hyperedge [25], and others. Furthermore, considering interactions across different systems, the cascading dynamics in interdependent or multilayer hypergraphs have also been

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investigated [26–28], which reveal that the system will be more vulnerable under the combined effect of intra-layer and inter-layer failure propagations.

Many studies have examined the effects of interdependencies on the cascading dynamics in multilayer networks, focusing on diverse aspects such as asymmetry [29], strength [30,31], assortativity [32,33], and redundancy [34] of interdependencies, and most of them has primarily concentrated on pairwise node-to-node interdependencies across layers [13,26]. However, higher-order interdependencies are more prevalent in real-word complex system, where groups of nodes in one layer are collectively interdependent on groups of nodes in another layer. In critical infrastructure, the smooth operation of a group of nodes, such as generation, transmission, and distribution, may require the proper function of another group of communication nodes, such as control and data acquisition nodes. Conversely, the groups of nodes in communication systems can also depend on several nodes in power grids [35]. This group-based interdependence can create more complicated failure dynamics than simple pairwise interdependencies. In gene regulation networks, groups of genes often act together to control the expression of other groups of genes or proteins [36]. These examples suggest that analyzing cascading dynamics with higher-order interdependencies across different systems can enhance the accuracy of theoretical models in reflecting the robustness of the complex system. Traditional methods focus only on pairwise connections across systems may overestimate or underestimate vulnerabilities of the complex systems as they do not consider higher-order cross-layer interdependencies. Incorporating these higher-order interdependencies may offer more insights into the robustness of multilayered systems, such as critical infrastructure, biological networks, and social systems, and therefore help design systems that are better equipped to withstand cascading failures by considering both intra-layer and inter-layer higher-order interactions.

In this paper, we investigate how higher-order inter-layer interdependencies affect the robustness and cascading dynamics in double-layer hypergraphs based on the percolation theory that has been used extensively in previous works [37–40]. To achieve this, we resort to double-layer hypergraphs to model the higher-order interdependencies across systems, incidentally also taking into account the higher-order interactions within each layer, which provides a more flexible framework for the study of cascading failures in complex systems. We identify three main findings from our study. First, as the average size of higher-order inter-layer interdependency groups increases, the system undergoes a transition from second-order to first-order phase transitions, highlighting the critical role that group size plays in the manner of the system disintegration. This result is especially important as it shows how the structure of interdependencies fundamentally alters the nature of phase transitions in a system. Second, for scale-free networks, we observe that larger interdependency groups not only trigger the disintegration manner of the system from a second-order phase transition to a first-order one, but can also cause two consecutive first-order transitions, implying a more complex and intricate response than previously recognized. This finding adds to the growing body of research by identifying a new phenomenon as well as its underlying mechanism in the cascading dynamics. Third, for scale-free and random double-layer hypergraphs, the effect of higher-order interdependence across layers on the robustness of the system is non-monotonic, with moderate-size groups leading to the weakest robustness, which suggests that the scale of the higher-order interdependencies has a major impact on the dynamical behavior of the system.

In the following sections, we outline our methodology, present our theoretical and numerical results, and discuss the broader implications of our findings for network design and system robustness.

2. Model

We construct a system composed of two layers of hypergraphs, A and B, where both layers have the same number N of nodes, and each node in one layer has a corresponding replica node in the other layer. In any given layer, nodes are connected through a total number M_c of connectivity hyperedges, which represents a group of nodes within a layer that are interconnected. At the same time, nodes in each layer may participate in a total number M_i of interdependency hyperedges, and each interdependency hyperedge has a counterpart in the other layer, representing groups of nodes in one layer that collectively depend on groups of nodes in the other layer. This modelling approach fundamentally differs from traditional one-to-one interdependencies of nodes [13], as it accounts for the collective and group-based nature of interactions rather than individual pairwise relationships. To simplify our study, we assume that each interdependency hyperedge in layer B contains all the replica nodes of the corresponding counterpart interdependency hyperedge in layer A, and vice versa. When some nodes fail due to attacks, some of the remaining nodes may become disconnected from the giant connected component of the hypergraph layer and fail as well. If all nodes within an interdependency hyperedge fail, this hyperedge fails and can no longer support its corresponding interdependency hyperedge in the other layer, leading to the failure of all nodes in its counterpart interdependency hyperedge.

The number of interdependency hyperedges that a node participates in is defined as the interdependency hyperdegree s, and the number of connectivity hyperedges connected to a node is defined as the connectivity hyperdegree k. In layer A, we denote the distribution of the interdependency hyperdegree as $\hat{P}^A(s)$ and the distribution of the connectivity hyperdegree as $P^A(k)$. Additionally, the number n of nodes that an interdependency hyperedge contains, i.e., the cardinality of the interdependency hyperedge, is distributed as $\hat{Q}^A(n)$, and the cardinality of a connectivity hyperedges is distributed as $Q^A(m)$. Similarly, nodes in layer B exhibit connectivity and interdependency hyperdegree distributions denoted as $P^B(k)$ and $\hat{P}^B(s)$, respectively. The cardinality distributions for interdependency hyperedges and connectivity hyperedges in layer B are $\hat{Q}^B(n)$ and $Q^B(m)$, respectively. In our study, no connectivity hyperdegree correlation is imposed between corresponding nodes in the two layers, i.e., the connectivity hyperdegree distributions of the two layers are independent. Based on the model definition, the average interdependency hyperdegree is given by $\langle s\rangle = \frac{\langle n\rangle M_i}{N}$, and similarly, the average connectivity hyperdegree is $\langle k\rangle = \frac{\langle m\rangle M_c}{N}$.

Cascading failures in the system are triggered by the initial removal of a fraction of nodes. Suppose a proportion $q = 1-p$ of nodes in layer A are randomly removed, and similarly, a proportion q of nodes in layer B are randomly removed. In each layer, the remaining nodes may fail due to two main mechanisms:

(1). Isolation failure: Nodes that are connected to the giant connected component via the removed nodes become isolated and subsequently fail.

(2). Interdependency failure: Nodes that are part of an interdependency hyperedge fail if its counterpart hyperedge in the other layer has collapsed.

The nodes are distributed in connectivity hyperedges in both hypergraphs in an entirely random way, and the nodes that initially fail in each layer are completely distinct. Failures caused by the first mechanism propagate through interdependency groups into the other layer, causing additional failures. These, in turn, trigger further failures due to isolation. This iterative process continues and the cascading failures occur until the system stabilizes, and eventually reaches a state where no further node or hyperedge failures occur (see Table 1 and Fig. 1 for an illustration of the cascading process).

To evaluate the structural integrity of the system, we use the final relative size $S_A \equiv G_A/N$ ($S_B \equiv G_B/N$) of the giant connected component in layer A (B), where $G_A(G_B)$ is the number of nodes in the

Table 1
Cascading failure process in double-layer hypergraph.

Step	Description
Step 1: Random node removal	Randomly remove a proportion $q = 1 - p$ of nodes from both layers A and B.
Step 2: Isolation failure	In each layer, check for nodes that have become isolated from the giant connected component. Isolated nodes fail.
Step 3: Interdependency failure	Identify interdependency hyperedges between the layers. If an interdependency hyperedge in one layer fails (due to node failure), the corresponding hyperedge and its internal nodes in the other layer may fail as well.
Step 4: Iterative process	Repeat steps 2-3 iteratively, allowing failures to propagate between the layers through a number of iterations.
Step 5: Stabilization	The process stops when no further node or hyperedge failures occur, and the system reaches a stable state.

giant connected component of layer $A(B)$. A layer with $S_A = 0$ ($S_B = 0$) is considered to be completely destroyed. To study the robustness of the system after removing a fraction $1 - p$ of nodes, we focus on the threshold p_c, above which the final hypergraph can have a state of incomplete destruction $S_A > 0$ or $S_B > 0$. Generally, a small critical point p_c suggests a more robust system, as it signifies that the giant connected component can persist after the cascading failure process, even when a significant fraction of nodes is initially removed.

This paper employs hypergraphs to describe the intra-layer higher-order interactions, where the hyperdegree distributions of nodes follow either scale-free or random distributions, and the cardinalities of both connectivity hyperedges and interdependency hyperedges are assumed to follow Poisson distributions. The hypergraphs are generated by the configuration model. Specifically, we first generate a hyperdegree sequence for the nodes $\{k_1, k_2, \dots, k_i, \dots, k_N\}$, which is ensured of following to either a power-law (scale-free) or Poisson distribution, and initially set M_c hyperedges as empty sets. Each node i is then randomly connected to k_i different hyperedges, with these k_i hyperedges including node i in their respective node sets. This approach ensures that the hyperdegree distribution of nodes conforms to the desired specification while simultaneously maintaining a Poisson distribution for the cardinalities of the hyperedges.

3. Theoretical and simulation results

3.1. Theory

To obtain the final relative size S^A of the giant connected component in layer A, we introduce an auxiliary variable R^A, which represents the probability that a random hyperedge reached by a random node can connect to the giant connected component of layer A. Similarly, S^B denotes the relative size of the giant connected component, and R^B represents the probability that a random hyperedge reached by a random node can connect to the giant connected component of layer B.

Additionally, for ease of expression, we introduce symbols to describe specific quantities. Firstly, the excess hyperdegree describes the number $k - 1$ of residual hyperedges possessed by a node with hyperdegree k reached by a random hyperedge. The distribution of the excess hyperdegree $k - 1$ for connectivity hyperedges is given by $P^A(k)k/\langle k\rangle$ and for interdependency hyperedges by $\hat{P}^A(s)s/\langle s\rangle$ in layer A. Secondly, the excess cardinality describes the number $m - 1$ of other nodes possessed by a random hyperedge with cardinality m of a random node, with a probability distribution of $Q^A(m)m/\langle m\rangle$ for connectivity hyperedges and $\hat{Q}^A(n)n/\langle n\rangle$ for interdependency hyperedges in layer A.

We also introduce generating functions for these distributions. $G^A_{k0}(x) = \sum_k P^A(k)x^k$ and $\hat{G}^A_{s0}(x) = \sum_s \hat{P}^A(s)x^s$ generate the hyperdegree distributions for connectivity and interdependency hyperedges, respectively. Furthermore, $G^A_{k1}(x) = \sum_k P^A(k)k/\langle k\rangle x^{k-1}$ and $\hat{G}^A_{s1}(x) = \sum_s \hat{P}^A(s)s/\langle s\rangle x^{s-1}$ generate the excess hyperdegree distributions for connectivity and interdependency hyperedges, respectively. Similarly, $G^A_{m0}(x) = \sum_m Q^A(m)x^m$ and $\hat{G}^A_{n0}(x) = \sum_n \hat{Q}^A(n)x^n$ denote the generating functions of cardinality distribution for connectivity and interdependency hyperedges, respectively. Generating functions for the excess cardinality distributions are $G^A_{m1}(x) = \sum_m Q^A(m)m/\langle m\rangle x^{m-1}$ and $\hat{G}^A_{n1}(x) = \sum_n \hat{Q}^A(n)n/\langle n\rangle x^{n-1}$ for connectivity hyperedges and interdependency hyperedges, respectively.

Similar generating functions are used for the corresponding quantities in layer B. Excess hyperdegree distributions in layer B are given by $P^B(k)k/\langle k\rangle$ for connectivity hyperedges and $\hat{P}^B(s)s/\langle s\rangle$ for interdependency hyperedges. The cardinality distributions of connectivity hyperedges and interdependency hyperedges are generated by $G^B_{m0}(x) = \sum_m Q^B(m)x^m$ and $\hat{G}^B_{n0}(x) = \sum_n \hat{Q}^B(n)x^n$, respectively. The excess cardinality distributions are generated by $G^B_{m1}(x) = \sum_m Q^B(m)m/\langle m\rangle x^{m-1}$ and $\hat{G}^B_{n1}(x) = \sum_n \hat{Q}^B(n)n/\langle n\rangle x^{n-1}$ for connectivity hyperedges and interdependency hyperedges, respectively.

We first examine the equation governing the probability R^A. For a randomly selected node in layer A, we choose one of its connectivity hyperedges at random. Assuming this hyperedge comprises m nodes, excluding the initially selected one, there remain $m - 1$ nodes within this hyperedge. The probability that the chosen connectivity hyperedge can connect to the giant connected component depends on the survival condition of the remaining $m-1$ nodes. The survival of any one of these nodes requires that the survival of all interdependency hyperedges involving its replica node in layer B. The survival condition for one random node in the remaining nodes can be divided into two cases:

(1). The replica of this remaining node enable to connect to the giant connected component of layer B. In this case, it requires at least one of its connectivity hyperedges to connect to the giant connected component. Considering the connectivity hyperdegree distribution $P^B(k)$ of the replica node, the probability that the node can connect to the giant connected component is given by $p[1 - \sum_k P^B(k)(1 - R^B)^k] = p[1 - G^B_{k0}(1 - R^B)]$, where $G^B_{k0}(x)$ is the generating function of the connectivity hyperdegree distribution of a random node in layer B.

(2). If the replica of this remaining node cannot connect to the giant connected component, then at least one other node in each of its interdependency hyperedges must survive. We assume that the replica node has s interdependency hyperedges, and the number s follows the distribution $\hat{P}^B(s)$. If the cardinality of a randomly selected interdependency hyperedge from the ones incident to that replica node is n, then the survival probability of this selected interdependency hyperedge is $1-(1-S^B)^{n-1}$. Since the probability of the selected interdependency hyperedge with size n is $\frac{n\hat{Q}^B(n)}{\langle n\rangle}$, the survival probability of the interdependency hyperedge is denoted as $\sum_n \frac{n\hat{Q}^B(n)}{\langle n\rangle}\left[1 - (1 - S^B)^{n-1}\right]$. Considering the hyperdegree distribution of interdependency hyperedges, we can obtain the survival probability of the replica node as: $\sum_s \hat{P}^B(s)\{\sum_n \frac{n\hat{Q}^B(n)}{\langle n\rangle}\left[1 - (1 - S^B)^{n-1}\right]\}^s$. In terms of generating functions, it can be simplified as $\hat{G}^B_{s0}(1 - \hat{G}^B_{n1}(1 - S^B))$.

Considering the above two scenarios, we can deduce the survival probability T^B of the replica node in layer B:

$$T^B = p\left(1 - G^B_{k0}(1 - R^B)\right) + \{1 - p\left[1 - G^B_{k0}(1 - R^B)\right]\}\hat{G}^B_{s0}\left(1 - \hat{G}^B_{n1}(1 - S^B)\right). \quad (1)$$

According to the probability T^B, we can obtain the probability R^A and the relative size S^A of the giant connected component. For a randomly chosen node in layer A, the number $m - 1$ of remaining nodes in one of its randomly selected connectivity hyperedges follow the distribution $\frac{Q^A(m)m}{\langle m\rangle}$. If any of these nodes can connect to the giant connected component, then the entire connectivity hyperedge

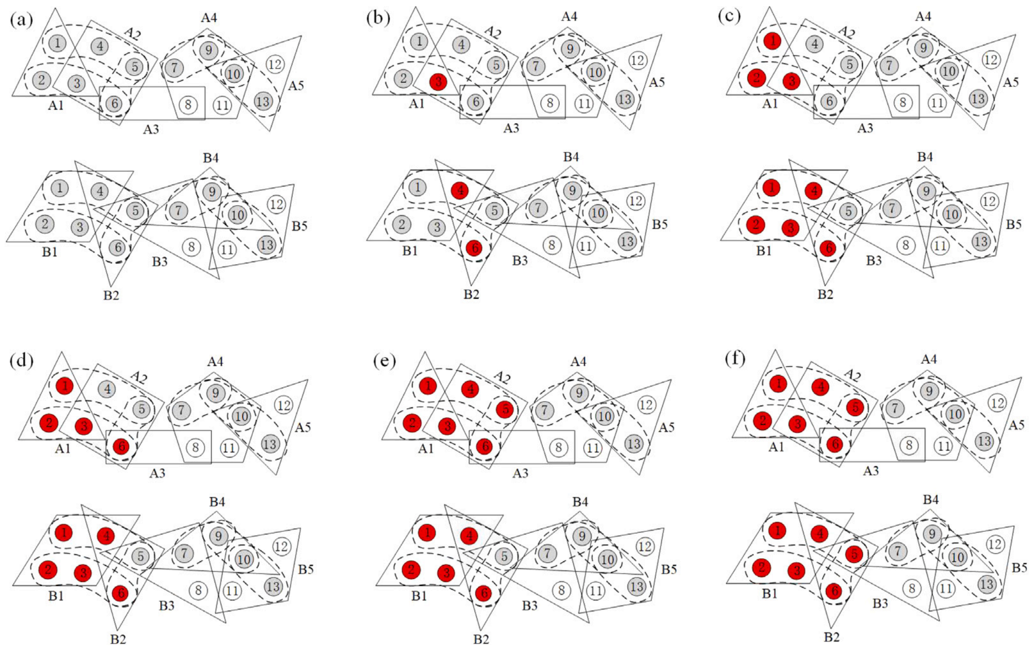

Fig. 1. Schematic diagram of the cascading failures in a double-layer hypergraph with higher-order inter-layer interdependencies, where the solid polygon-enclosed node sets represent connectivity hyperedges, while the dashed arc-enclosed node sets represent interdependency hyperedges. (a) The intact system before attack. (b) The red nodes indicate failed nodes: node 3 in layer A and nodes 4 and 6 in layer B. (c) Nodes 1 and 2 in layer A, and nodes 1, 2, and 3 in layer B, fail by disconnecting from the giant connected component. (d) The failure of the interdependency hyperedge $\{2,3,6\}$ in layer B causes the failure of the corresponding hyperedge $\{2,3,6\}$ in layer A. (e) The failure of node 6 in layer A leads to the failure of nodes 4 and 5. (f) The failure of the interdependency hyperedge $\{1,4,5\}$ in layer A causes the failure of the corresponding hyperedge $\{1,4,5\}$ in layer B, and the system reaches to a stable state.

can connect to the giant connected component. The number $k-1$ of remaining connectivity hyperedges for the randomly selected node follows the distribution $\frac{P^A(k)k}{\langle k\rangle}$, and the probability that this node can connect to the giant connected component is $1-\sum_k \frac{P^A(k)k}{\langle k\rangle}(1-R^A)^{k-1}$. Additionally, considering the probability p that this node is initially reserved, as well as the probability T^B that its replica node is preserved, we derive the self-consistent equation governing the probability R^A

$$R^A = 1 - G^A_{m1}\left(1-p\left[1-G^A_{k1}(1-R^A)\right]T^B\right). \tag{2}$$

Similarly, we can derive that S^A satisfies the following equation:

$$S^A = p\left(1-G^A_{k0}(1-R^A)\right)T^B. \tag{3}$$

By a completely analogous process, we can derive the equations governing the variables T^A, R^B, and S^B as follows:

$$\begin{cases} T^A = p\left(1-G^A_{k0}(1-R^A)\right)+\left[1-p(1-G^A_{k0}(1-R^A))\right]\hat{G}^A_{s0}(1-\hat{G}^A_{n1}(1-S^A)),\\ R^B = 1-G^B_{m1}\left(1-p(1-G^B_{k1}(1-R^B))T^A\right),\\ S^B = p\left(1-G^B_{k0}(1-R)\right)T^A. \end{cases} \tag{4}$$

For simplicity, this paper assumes that the hypergraph layers A and B share identical distributions for connectivity hyperdegree, interdependency hyperdegree, cardinality of connectivity hyperedges and cardinality of interdependency hyperedges, i.e., $P^A(k)=P^B(k):=P(k)$, $\hat{P}^A(s)=\hat{P}^B(s):=\hat{P}(s)$, $Q^A(m)=Q^B(m):=Q(m)$, and $\hat{Q}^A(n)=\hat{Q}^B(n):=\hat{Q}(n)$. Given the symmetry between layers A and B, it is evident that $T^A=T^B:=T$, $S^A=S^B:=S$ and $R^A=R^B:=R$, and the equations can be simplified as:

$$T = p(1-G_{k0}(1-R))+\left[1-p(1-G_{k0}(1-R))\right]\hat{G}_{s0}(1-\hat{G}_{n1}(1-S)). \tag{5}$$

According to the probability T, we can obtain the probability R and the relative size S of the giant connected component

$$\begin{cases} R = 1-G_{m1}(1-p(1-G_{k1}(1-R))T) := F_1(p,R,S),\\ S = p(1-G_{k0}(1-R))T := F_2(p,R,S). \end{cases} \tag{6}$$

The equation $F_1(p,R,S)=R$ exhibits bifurcation behavior, where the line $R=0$ represents the trivial solution, while a nontrivial solution emerges at the bifurcation point, as shown in Fig. 2. This phenomenon significantly impacts the phase transition behavior of the system. Typically, as the initial reserve probability p of nodes increases, the giant connected component of the system may emerge through either a first-order or second-order phase transition. The critical points of these transitions can be determined by studying Eq. (6). When the system undergoes a second-order phase transition, the variables S and R continuously emerge from zero at the critical point p_c^{II}, as shown in Fig. 2(a–c). To elucidate this behavior, we perform a linear expansion of Eq. (6) at the point $(R=0, S=0)$, resulting in the following equations:

$$\begin{cases} R = F_1(p_c^{II},0,0) + \left.\dfrac{\partial F_1}{\partial R}\right|_{R=0,S=0} dR + \left.\dfrac{\partial F_1}{\partial S}\right|_{R=0,S=0} dS,\\ S = F_2(p_c^{II},0,0) + \left.\dfrac{\partial F_2}{\partial R}\right|_{R=0,S=0} dR + \left.\dfrac{\partial F_2}{\partial S}\right|_{R=0,S=0} dS. \end{cases} \tag{7}$$

This formulation allows us to analyze the behavior of the system near the critical point. The critical point p_c^{II} is precisely identified when the determinant of the Jacobian matrix $det(J-I)=0$, which equates to setting the conditions that lead to nontrivial solutions of the linearized system. The Jacobian matrix J is

$$J = \begin{pmatrix} p_c^{II}\cdot\hat{G}_{s0}(0)\cdot G'_{k1}(1)\cdot G'_{m1}(1) & 0\\ p_c^{II}\cdot\hat{G}_{s0}(0)\cdot G'_{k0}(1) & 0 \end{pmatrix}. \tag{8}$$

Subsequently, we deduce that the critical point for a second-order phase transition, p_c^{II}, satisfies the following condition:

$$p_c^{II}\cdot\hat{G}_{s0}(0)\cdot G'_{k1}(1)\cdot G'_{m1}(1) = 1. \tag{9}$$

Considering that the cardinality of the interdependency hyperedges follows the Poisson distribution, the second-order percolation transition point p_c^{II} is

$$p_c^{II} = \frac{\langle k\rangle}{\langle k(k-1)\rangle}\cdot\frac{1}{\langle m\rangle}e^{\langle s\rangle}. \tag{10}$$

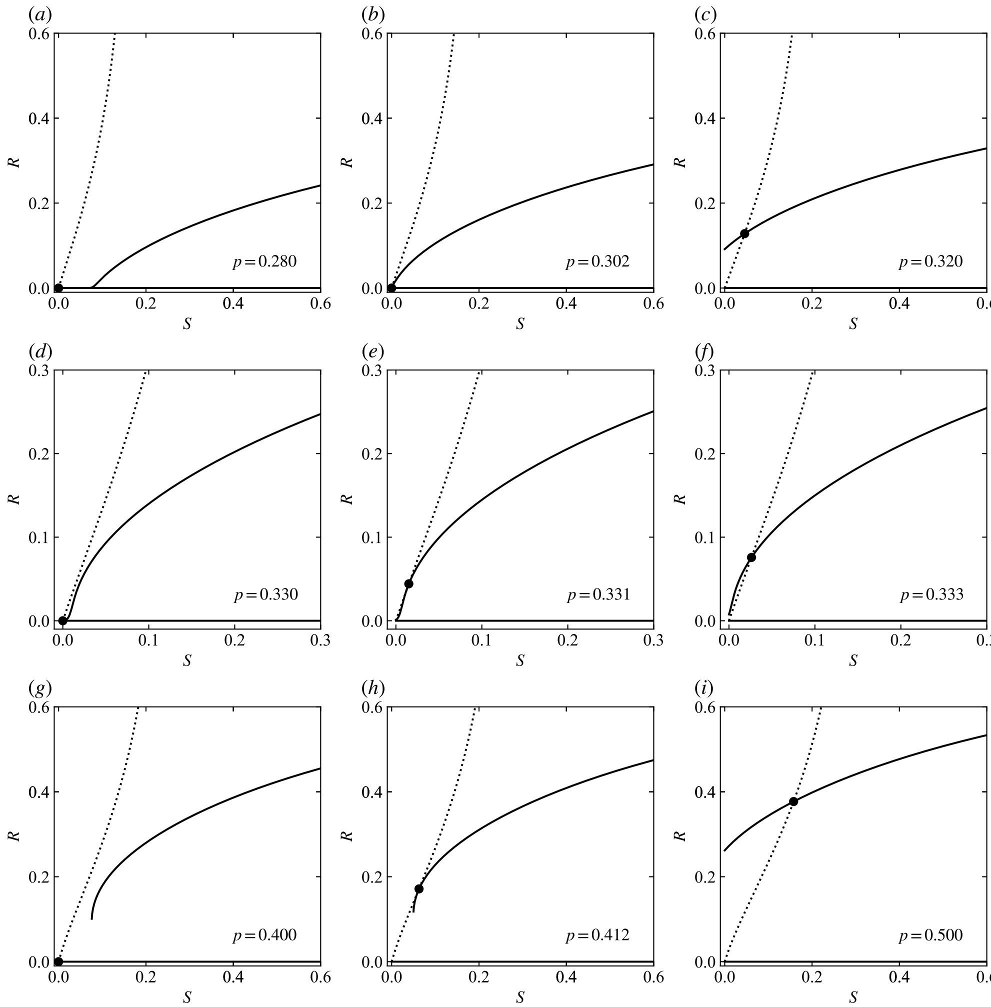

Fig. 2. The function curves of Eq. (6) for double-layer random hypergraphs with different values of p, where the solid line denotes the curve defined by $F_1(p,R,S)=R$, dotted line is the function $F_2(p,R,S)=S$, and the black dots are the solutions. (a–c) the results for $\langle m\rangle=3$ and $\langle n\rangle=1.0$; (d–f) the results for $\langle m\rangle=3$ and $\langle n\rangle=1.1$; (g–i) the results for $\langle m\rangle=3$ and $\langle n\rangle=1.5$. In all panels, we set $\langle s\rangle$ to be the same as $\langle n\rangle$ and $\langle k\rangle$ to be the same as $\langle m\rangle$.

The moments $\langle k(k-1)\rangle$ capture the heterogeneity of hyperdegree distribution, which can significantly affect the robustness of the double-layer hypergraph. When the second-order moments diverge, the system exhibits very strong robustness to random attacks, and the percolation threshold p_c^{II} approaches zero. The percolation threshold p_c^{II} is inversely related to the average cardinality $\langle m\rangle$ of connectivty hyperedges, meaning that as $\langle m\rangle$ increases, the system's robustness improves, and the percolation threshold p_c^{II} becomes smaller. The term $e^{\langle s\rangle}$ captures the effect of interdependency, where a larger average cardinality of interdependency hyperedges increases the likelihood of cascading failures. When $\langle s\rangle=0$, the system reduces to a standard percolation on hypergraphs [26].

For a first-order phase transition, the condition arises where the curves $F_1(p,R,S)=R$ and $F_2(p,R,S)=S$ are tangent at a point, as illustrated in Fig. 2(d–f) or (g–i). At this juncture, designated as p_c^I, the solution of the equations jumps from (R_c,S_c) to $(0,0)$. During such a transition, it is crucial to identify p_c^I, R_c, and S_c such that the equation $\partial F_1/\partial R\partial F_2/\partial S=\partial F_1/\partial S\partial F_2/\partial R$ holds true. This yields the solution to the equations $F_1(p,R,S)=R$ and $F_2(p,R,S)=S$ at the point of a first-order phase transition.

If the system changes from a first-order phase transition to a second-order phase transition during a collapse, the conditions for both first-order and second-order phase transitions are simultaneously satisfied. To analyze this behavior, we perform a Taylor expansion of Eq. (6) around the critical point $p=p_c^{II}$, i.e., $(S=0,R=0)$. Since the condition $det(J-I)=0$ is already satisfied (implying the second-order phase transition condition), we now focus on the second-order term in the Taylor expansion, and we can have

$$\left.\frac{\partial^2 F_1}{\partial R\partial S}\right|_{R=0,S=0} dR\,dS+\frac{1}{2}\left(\left.\frac{\partial^2 F_1}{\partial R^2}\right|_{R=0,S=0} d^2R+\left.\frac{\partial^2 F_1}{\partial S^2}\right|_{R=0,S=0} d^2S\right)=0. \tag{11}$$

Based on the derivative relations in Eq. (7), we establish the relationship $dR=\frac{\langle k(k-1)\rangle}{\langle k\rangle^2}\langle m\rangle dS$. By substituting the second-order phase transition point p_c^{II} into this equation, we derive that:

$$\frac{\langle n\rangle}{\langle m\rangle}-\left(\frac{e^{\langle s\rangle}}{\langle m\rangle}-\frac{e^{2\langle s\rangle}}{\langle m\rangle}+\frac{1}{2}\frac{\langle k(k-1)(k-2)\rangle}{\langle k\rangle^2}+\frac{1}{2}\frac{\langle k(k-1)\rangle}{\langle k\rangle^2}\right)=0. \tag{12}$$

Given the hyperdegree distribution of the hypergraph layers for both conectivity hyperedges and interdependency hyperedges, as well as the average cardinality $\langle m\rangle$ of the connectivity hyperedges, we can use the above equation to determine the critical point $\langle n\rangle_c$ of the interdependency hyperedges where the percolation transition of the system changes from a first-order to a second-order phase transition.

3.2. Random hypergraphs

In the following of our research, we will fix the number of interdependency hyperedges and connectivity hyperedges in both layers

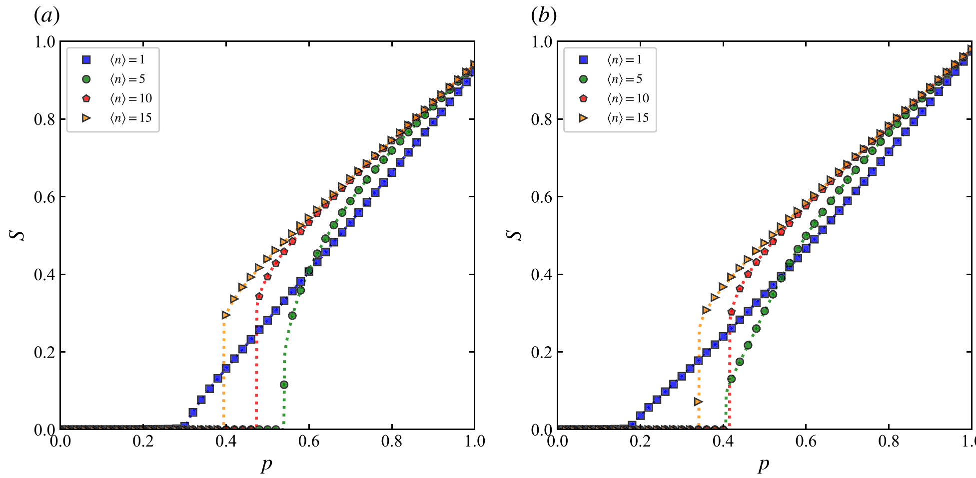

Fig. 3. The percolation transition for different average cardinality $\langle n\rangle$ of interdependency hyperedges for double-layer random hypergraphs. (a–b) The final relative size S of the giant component as a function of the node reserving probability p for average cardinality of connectivity hyperedges, $\langle m\rangle = 3$ and $\langle m\rangle = 4$, respectively. In both panels, we set $\langle s\rangle$ to be the same as $\langle n\rangle$ and $\langle k\rangle$ to be the same as $\langle m\rangle$. The symbols represent simulation results that obtained over 50 independent runs for the system size $N = 10^5$, and dotted lines represent theoretical predictions.

to be equal to the number N of nodes. Under this condition, the average size of interdependency hyperedges, $\langle n\rangle$, will be equal to the average hyperdegree of nodes for interdependency hyperedges, $\langle s\rangle$, and the average hyperdegree of nodes for connectivity hyperedges, $\langle k\rangle$, will be the same as the average size of connectivity hyperedges, $\langle m\rangle$. Fig. 3 illustrates the size of the giant connected components in double-layer hypergraphs as p increases, employing Poisson cardinality and connectivity hyperdegree distributions. We observe distinct behaviors in the emergence of the giant connected components depending on the average cardinality $\langle n\rangle$ of interdependency hyperedges. When $\langle n\rangle$ is small, the giant connected component emerges continuously as p exceeds a percolation transition point, denoted as p_c^{II}, such as when $\langle n\rangle = 1$ depicted in Fig. 3(a) or (b). Conversely, for larger $\langle n\rangle$, the giant connected component appears discontinuously at the percolation transition point p_c^I, indicating a first-order phase transition, applicable for $\langle n\rangle \geq 5$, as shown in Fig. 3(a) or (b). Additionally, we note that with increasing $\langle n\rangle$, the critical thresholds p_c for system fragmentation, p_c^{II} or p_c^I, initially increase and then decline. These results highlight the significant impact that the average size of interdependency groups across layers has on the fragmentation pattern and the robustness of the system when suffering random attack. Specifically, an intermediate average size of interdependency groups results in the poorest robustness. These observations suggest that the average cardinality of interdependency groups plays a crucial role in determining the system's response to disruptions, influencing both the robustness and fragmentation patterns of the double-layer system.

Fig. 4 shows the influence of the average cardinality of interdependency hyperedges $\langle n\rangle$ on the phase transition point of system fragmentation. It reveals that as $\langle n\rangle$ increases, the fragmentation manner of the system shifts from a second-order to a first-order phase transition, with the phase transition point first increasing and then decreasing. This indicates that $\langle n\rangle$ not only affects the fragmentation manner of the system but also influences its robustness. As the interdependency strength of the system increases, so does the sensitivity to perturbations, making the system more susceptible to abrupt changes in structure and function under random attack. Higher values of $\langle n\rangle$ tend to correlate with a more brittle system that is prone to sudden collapses rather than gradual degradation.

3.3. Scale-free hypergraphs

Fig. 5 provides the relative size of the giant connected component versus p in the final cascading dynamics of a double-layer scale-free

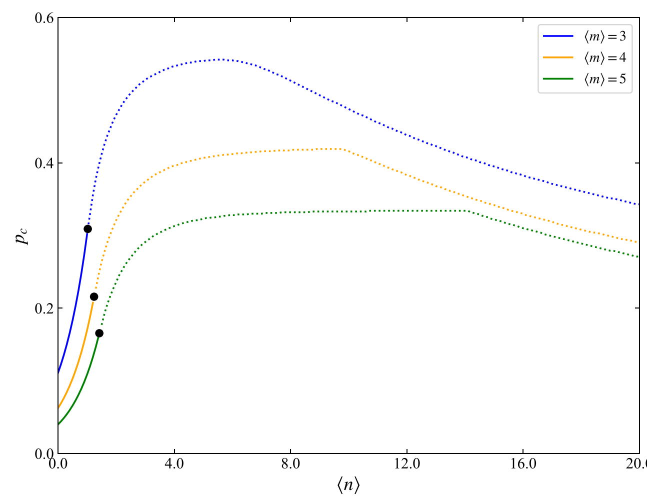

Fig. 4. Theoretical results for the relationship between the phase transition point p_c and the average cardinality $\langle n\rangle$ of interdependency hyperedges for double-layer random hypergraphs with average connectivity hyperedge cardinality $\langle m\rangle = 3, 4, 5$. Here we set $\langle s\rangle$ to be the same as $\langle n\rangle$ and $\langle k\rangle$ to be the same as $\langle m\rangle$. Black dots indicate the critical points where the system transitions from a second-order phase transition (solid lines) to a first-order phase transition (dotted lines).

hypergraph. In this model, the connectivity hyperdegree distributions of both layers follow a power-law distribution $P(k) \sim k^{-\gamma}$, with k in the range $[k_{min}, k_{max}]$, and the cardinality distributions of both connectivity and interdependency hyperedges follow Poisson distributions. As the probability of initial node retention increases, we observe significant changes in the size of the giant connected component. Notably, the nature of the phase transition is influenced by the average cardinality $\langle n\rangle$ of the interdependency hyperedges.

For lower values of $\langle n\rangle$, the system experiences second-order phase transitions, with gradual changes in the size of the giant connected component. However, as $\langle n\rangle$ increases, the transition becomes abrupt, signaling a shift to first-order phase transitions. Interestingly, for larger values of $\langle n\rangle$, the system may undergo two consecutive first-order phase transitions, revealing complex interactions and interdependencies within the system that significantly affect its stability and robustness. As $\langle n\rangle$ continues to increase, the system will become more fragile, and the second first-order phase transition point will become smaller, resulting in the reduction of two first-order phase transitions

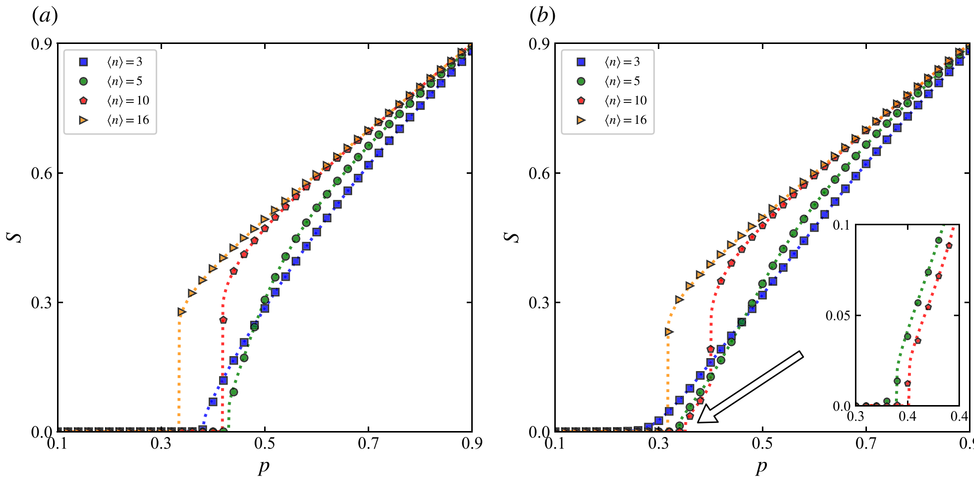

Fig. 5. The percolation transition for different average cardinality $\langle n\rangle$ of interdependency hyperedges for double-layer scale-free hypergraphs. (a) The final relative size S of the giant connected component as a function of the reserving probability p for $\langle m\rangle = 4$. Both the hypergraph layer follows the power-law hyperdegree distribution, where the power-law exponent $\gamma = 2.6$, $k_{min} = 2$ and $k_{max} = 132$. (b) The results for $\langle m\rangle = 5$, where the power-law exponent $\gamma = 2.3$, $k_{min} = 2$ and $k_{max} = 128$. In both panels, we set $\langle s\rangle$ to be the same as $\langle n\rangle$ and $\langle k\rangle$ to be the same as $\langle m\rangle$. The symbols represent simulation results that obtained over 50 independent runs for the system size $N = 10^5$, and dotted lines represent theoretical predictions. The inset highlights the first first-order phase transition.

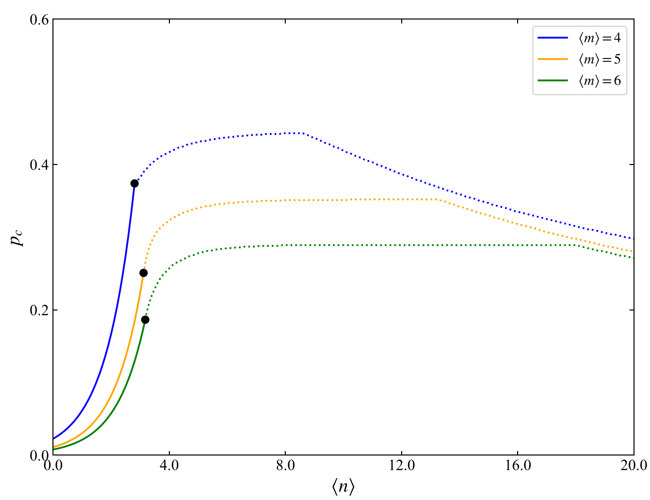

Fig. 6. Theoretical results for the relationship between the phase transition point p_c and the average cardinality $\langle n\rangle$ of interdependency hyperedges for double-layer scale-free hypergraphs with average connectivity hyperedge cardinality $\langle m\rangle = 4, 5, 6$, where the parameters $(\gamma, k_{min}, k_{max})$ are set to $(2.6, 2, 132)$, $(2.3, 2, 128)$ and $(2.1, 2, 109)$, respectively. Here we set $\langle s\rangle$ to be the same as $\langle n\rangle$ and $\langle k\rangle$ to be the same as $\langle m\rangle$. Black dots indicate the critical points where the system transitions from a second-order phase transition (solid lines) to a first-order phase transition (dashed lines).

into a single first-order phase transition, i.g., $\langle n\rangle = 16$ for the case $(\gamma, k_{min}, k_{max}) = (2.3, 2, 128)$ as shown in Fig. 5(b).

Moreover, as $\langle n\rangle$ increases, the percolation threshold initially rises, but then decreases, as shown in Fig. 6, which suggests that the average cardinality of interdependency hyperedges not only influences the type of phase transition but also has a non-monotonic effect on the system's robustness. This implies that intermediate values of $\langle n\rangle$ may lead to the weakest system robustness, as indicated by the highest critical point of percolation, whereas too high or too low values of $\langle n\rangle$ may enhance the ability of the system to resist the external perturbations.

To advance our understanding of the phenomenon of consecutive first-order phase transitions in double-layer scale-free hypergraphs, we analyze the bifurcation behavior of $F_1(p, R, S) = R$. At lower values of the average cardinality $\langle n\rangle$, the curve defined by $F_1(p, R, S) = R$ undergoes a continuous bifurcation at a point $(S'_c, 0)$. The bifurcation point can be determined by $\frac{\partial F_1}{\partial R}\Big|_{R=0,S=S'_c} = 1$, which is

$$p \cdot \hat{G}_{s0}\left(1 - \hat{G}_{n1}(1 - S'_c)\right) \cdot G'_{k1}(1) \cdot G'_{m1}(1) = 1. \tag{13}$$

As p increases to p_c^{II}, S'_c gradually approaches 0, and the curve of $F_1(p, R, S) = R$ intersects with the curve of $F_2(p, R, S) = S$, leading to the emergence of the giant connected component in the form of a second-order phase transition. In this case, the solutions for Eq. (6) change continuously and smoothly, characteristic of second-order phase transitions, as shown in Fig. 7(a–c).

As $\langle n\rangle$ exceed the critical value of the transition point of the first-order and the second-order phase transition point determined by Eq. (12), the curves of $F_1(p, R, S) = R$ and $F_2(p, R, S) = S$ can be tangent at $p = p_c^I$, leading to the emergence of the giant connected component in the manner of a discontinuous first-order phase transition, as shown in Fig. 7(d–f).

As $\langle n\rangle$ further increases, before the emergence of the giant connected component, the curve of $F_1(p, R, S) = R$ can bifurcate at a point (S'_c, R'_c) in a discontinuous manner. When the bifurcation manner changes from a continuous transition to a discontinuous one, the amplitude R'_c of discontinuous jumps approaches 0. In this situation, we perform a Taylor expansion of $F_1(p, R, S) = R$ at the bifurcation point, and obtain:

$$F_1(p, R'_c, S'_c) + \frac{\partial F_1}{\partial R}\Big|_{R=R'_c,S=S'_c} dR + \frac{\partial^2 F_1}{\partial R^2}\Big|_{R=R'_c,S=S'_c} d^2R = R. \tag{14}$$

Considering Eq. (13) and imposing $R'_c = 0$, the above equation is

$$\begin{aligned} &p^2\left(\hat{G}_{s0}\left(1 - \hat{G}_{n1}(1 - S'_c)\right)\right)^2 G'_{k1}(1)^2 G''_{m1}(1) \\ &- 2p^2\left[-\hat{G}_{s0}\left(1 - \hat{G}_{n1}(1 - S'_c)\right) G'_{k0}(1) + G'_{k0}(1)\right] G'_{k1}(1) G'_{m1}(1) \\ &+ p\hat{G}_{s0}\left(1 - \hat{G}_{n1}(1 - S'_c)\right) G''_{k1}(1) G'_{m1}(1) = 0, \end{aligned} \tag{15}$$

which can be further simplified as

$$2p^2\langle k(k-1)\rangle\langle m\rangle - 2p\langle k\rangle - \frac{\langle k(k-1)(k-2)\rangle}{\langle k(k-1)\rangle} - 1 = 0. \tag{16}$$

According to Eq. (16), we can determine the critical value p'_c. When $p > p'_c$, R emerges abruptly as S increases to the critical value S'_c with the constraint of $F_1(p, R, S) = R$. However, for $p \le p'_c$, R emerges continuously. If p'_c does not lead to the emergence of a nontrivial solution of Eq. (6), as p increases, the bifurcation point (S'_c, R'_c) of $F_1(p, R, S) = R$ can contact the curve $F_2(p, R, S) = S$, causing the solutions of Eq. (6) to abruptly change and triggering a first-order

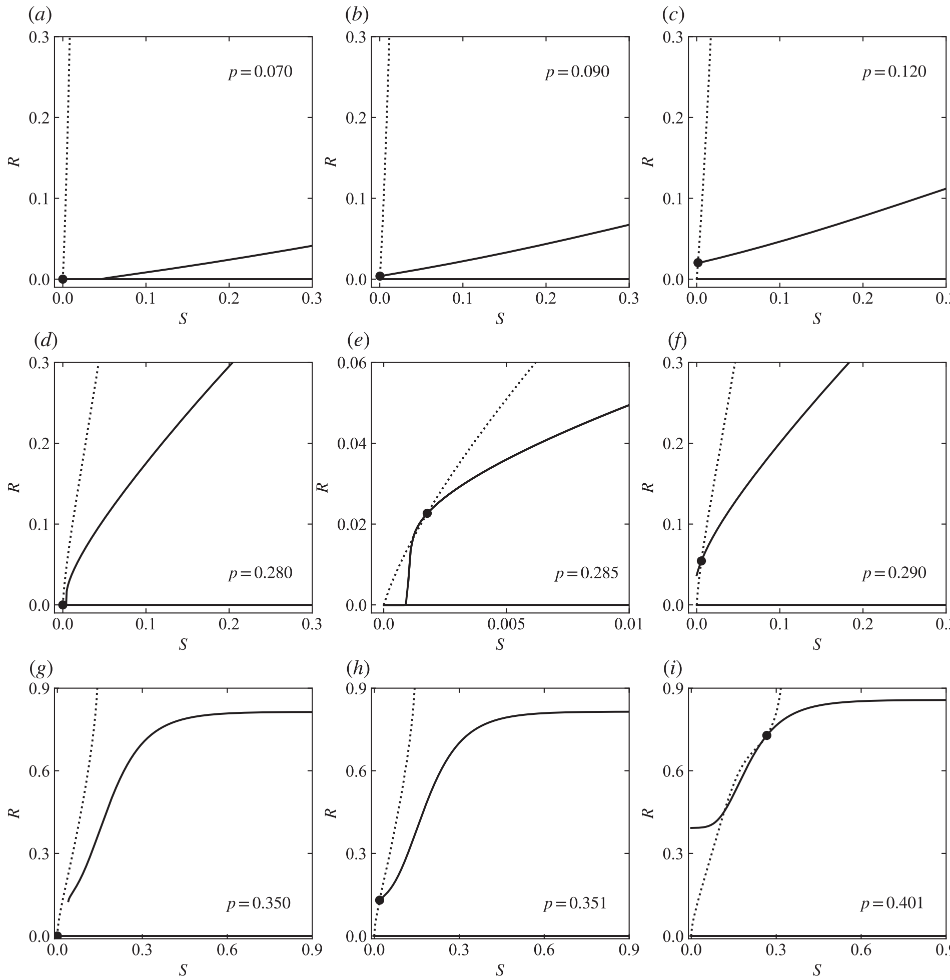

Fig. 7. Illustration of the phase transitions for $F_1(p,R,S)=R$ (solid line) and $F_2(p,R,S)=S$ (dashed line)for double-layer scale-free hypergraphs, where the average hyperdegree $\langle k\rangle$ is set to 5 with $(\gamma,k_{min},k_{max})=(2.3,2,128)$. Panels (a–c) show the second-order phase transition, where the average cardinality of interdependency hyperedges $\langle n\rangle=2.0$. Panels (d–f) show a first-order phase transition, where $\langle n\rangle=3.3$. (g–i) depict the occurrence of two consecutive first-order phase transitions, where $\langle n\rangle=10$. In all panels, we set $\langle s\rangle$ to be the same as $\langle n\rangle$ and $\langle m\rangle$ to be the same as $\langle k\rangle$.

phase transition in the system, as shown in Fig. 7(h). Additionally, with further increases in p, the equations $F_1(p,R,S)=R$ and $F_2(p,R,S)=S$ may become tangent, resulting in the other one first-order phase transition in the system, as shown in Fig. 7(i).

4. Conclusion

In this study, we explored how higher-order interdependencies across layers influence the robustness and cascading dynamics in double-layer hypergraphs. By incorporating interdependency hyperedges that account for higher-order interdependencies across layers, we established a more general framework and the corresponding theory for understanding cascading dynamics in multilayer systems with random and scale-free hyperdegree distributions. Our study reveals key insights, such as the impact of interdependency group size on the natrue of phase transitions and system resilience, as well as the non-monotonic effects of this size on system robustness.

Firstly, we showed that the system shifts its disintegration manner from second-order to first-order phase transitions as the average size of higher-order inter-layer interdependency groups increases, which highlights the non-negligible role of group size in determining the nature of phase transitions and system resilience. Specifically, in the case of double-layer scale-free hypergraphs, enlarging the size of these interdependency groups can not only induce both second- and first-order phase transitions but might also result in two successive first-order transitions. This result is particularly innovative because previous studies have generally focused on the coexistence of first- and second-order phase transitions. In contrast, observing successive first-order transitions is rare. We have also explained that these two first-order phase transitions originate from different underlying mechanisms within the system. The first first-order phase transition occurs specifically in scale-free hypergraphs and is driven by their inherent hyperdegree heterogeneity. As the interdependency group size increases, high-hyperdegree hub nodes become increasingly involved in multiple interdependency groups, and the failure of a single interdependency hyperedge directly leads to the failure of the associated hub node. These interdependency hyperedges fail primarily due to their counterparts in the other hypergraph layer, which often include low-hyperdegree nodes, and these nodes are more susceptible to disconnecting from the giant connected component. As the hub nodes play critical roles in maintaining the overall connectivity, their failures can lead to an abrupt reduction in the size of the giant connected component with the increase of the fraction of initial removed nodes, which is the reason why the first first-order phase transition occurs. The second first-order phase transition occurs due to the expansion of the interdependency group size. As the size of these interdependency groups increases, the range and magnitude of risk propagation across the hypergraph also expand, which can lead to a sudden, large-scale collapse of the system when a critical threshold of removed nodes is reached. This mechanism is robust and observed across both random and scale-free hypergraphs. It is primarily driven by the interconnected nature of interdependency groups, which amplifies the spread of failures as their sizes increase.

Secondly, we found that the effects of higher-order inter-layer interdependencies on system robustness are non-monotonic. Both scale-free and random hypergraphs show their weakest robustness at intermediate group sizes, marked by the highest critical point for system fragmentation, which indicates that the size of interdependency groups has a dual effect: small and large groups enhance robustness, while intermediate sizes are the most detrimental. This phenomenon arises because the average number of interdependency groups a node participates in increases with the group size. While larger interdependency groups are less likely to fail due to the robustness of their counterparts in the other layer, the increased participation of nodes in multiple interdependency groups introduces vulnerabilities. Specifically, a node fails if any of the interdependency groups it belongs to fails, amplifying the cascading effect. Thus, the average size of interdependency groups exerts a dual effect on system robustness: small and large groups tend to enhance robustness, while intermediate sizes maximize the system's susceptibility to cascading failures. This interplay underlies the observed non-monotonic behavior. From a practical perspective, this finding suggests that both overly concentrated and overly diffuse interdependencies should be avoided in infrastructure networks such as power grids or communication systems. Smaller and larger groupings can either localize failures or distribute interdependencies more effectively, and both scenarios contribute to enhancing the robustness of the system. However, intermediate-sized interdependency groups can create the most disruptive cascading failures, which implies the need for targeted interventions in the scale of the interdependency groups in the system. Moreover, as the average interdependency hyperedge size increases, although the critical point for collapse decreases, the system experiences a first-order phase transition, leading to a more abrupt collapse. This insight is crucial for designing early warning systems and improving failure mitigation strategies in real-world systems with varying levels of interdependency.

Our findings offer valuable insights for designing more resilient networked infrastructures, particularly in systems where higher-order interdependencies across layers are unavoidable, which differs from previous works focusing on intra-layer higher-order interactions [27,41–43] or pairwise inter-layer connections [13,29,31]. Our research highlights the importance of group-based higher-order interdependencies across layers in shaping the system's vulnerability to cascading failures and emphasizes that higher-order cross-layer dynamics require new approaches to understanding and mitigating system fragility. Despite this, there are certain limitations to the current model that should be acknowledged. One key limitation is the assumption that the sizes of pairs of interdependency groups across the two layers are identical. This assumption simplifies the model and theory, but it may not fully reflect the asymmetries observed in real-world systems, where pairs of interdependency groups can vary in size due to factors such as resource constraints or differences in system functionality. Additionally, our model has not considered how the dynamic changes of interdependency groups over time could affect the cascading dynamics. Thus, in future research, extending the model to integrate these factors would improve its applicability to more specific contexts and could provide more understanding of the robustness and cascading failures of multilayer networks.

In conclusion, the study of higher-order interdependencies in multilayer hypergraphs provides new perspectives on the modelling of cascading dynamics in complex systems and establishes a theoretical basis for the design of systems that can better resist cascading failures by tuning the heterogeneity of hyperdegree distribution or the size of interdependency groups. Moreover, the proposed model is also helpful in developing more effective control strategies for real-world complex systems to enhance their resilience or suppress cascading failures.

CRediT authorship contribution statement

Chun-Xiao Jia: Writing – review & editing, Visualization, Validation, Investigation, Conceptualization. **Run-Ran Liu:** Writing – original draft, Visualization, Resources, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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