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A threshold model of cascading failure on random hypergraphs

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ABSTRACT

Higher-order interactions are ubiquitous in the real world and play a critical role in maintaining the overall function of complex systems. To investigate the effects of higher-order interactions on cascading dynamics, we propose a threshold model of cascading failure on hypergraphs that describes the propagation mechanism of failures among nodes and hyperedges. We assume that a hyperedge fails when the fraction of failed nodes within the hyperedge exceeds a specified threshold. Additionally, once a hyperedge fails, all its remaining nodes also fail. Through numerical simulations and theoretical analysis, we reveal a dual effect of hyperedges on the robustness of hypergraphs: they can not only strengthen the connections among nodes and promote the emergence of giant components in the hypergraph but also increase the risk of failure transmission among nodes and debilitate the hypergraph. Our work provides a theoretical framework for understanding the cascading failure of complex systems with high-order interactions and offers a useful tool for designing robust complex systems with such interactions.

1. Introduction

Robustness refers to the resistance of a system to external attacks or disturbances. Due to the interactions or interdependencies among constituent units, complex systems often suffer cascading failures when attacked [1,2]. Both modeling the cascading dynamics and boosting the robustness of complex systems are long-term and significant scientific problems. Over the past two decades, many cascading models have been proposed based on network representations of complex systems, such as single networks [3–5] or multilayer networks [6–9]. In most existing models, pairwise interaction is used to describe the coupling or interdependency of one node to another. However, it has been widely recognized that interactions between units may go beyond pairwise interactions in many biological [10], physical [11], and social systems [12,13], such as multiple protein interactions [14], social collaborations [15], and species interactions in ecosystems [10]. Thus, it is of great theoretical and general significance to establish a cascading model based on higher-order interactions to investigate the robustness of complex systems.

Hypergraphs or hypernetworks are often used as an effective tool to characterize higher-order interactions, where hyperedges consist of multiple nodes [16]. This allows a set of interacting nodes to be represented by a hyperedge [10,17]. In some technological, social, or biological systems, a group of nodes work together to form a functional module, where the failure of a fraction of members usually may not destroy the function of the entire group, i.e., a hyperedge may exist even after some nodes fail due to the fault tolerance mechanism. This feature is indescribable by pairwise interactions since a link has only two nodes in simple networks, the loss of one will cause the failure of the entire connection. In addition, the group interaction can also describe the microscopic mechanism of a hyperedge failure caused by its members. For example, in a power system, some power stations or substations work in parallel, and the whole system may need to be overhauled in the case of one or more failures. Therefore, studying the cascading dynamics of complex networks under higher-order interactions can provide a foundation for more abundant microscopic interaction mechanisms, helping to reveal the occurrence mechanisms of cascading failures in complex systems and the key factors affecting their robustness.

The cascading dynamics on complex networks with pairwise interactions have received extensive attention in terms of diverse microscopic propagation mechanisms of failures, and have uncovered a series of critical phenomena and macroscopic characteristics of failure propagation [18]. The threshold model is the most straightforward way to describe the interaction mechanism among nodes, where a node will fail immediately if the fraction of failed nodes in its neighbors exceeds a threshold [1]. The threshold model has been extensively used to study the cascading dynamics in various types of networks. One of the advantages of the threshold model is that it can capture the heterogeneity of nodes in the network, which is an important factor that affects the

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spread of failures [19,20]. Moreover, the threshold model has been used to study the effects of degree correlation and modularity features of networks on the spread of failures [21], multiplexity-facilitated cascades [22] and response heterogeneity in multiplex networks [23]. At the same time, some critical phenomena and system-level properties of failure propagation have been revealed, such as the criteria for the global cascade [5,24], change in phase transition type [25], and two-tiered structure in cascading process [26]. Furthermore, the traditional threshold model has been extended to hypergraphs and studied the condition of large cascades, where node activation occurs if the number of active neighbors connected by hyperedges surpasses a threshold [27]. However, previous research mainly focused on the influence of neighboring node states on individual node states, neglecting the consideration of how the states of constituent nodes impact the functionality of system modules. To address this, we generalize the threshold model to incorporate higher-order interactions, considering both node-to-edge and edge-to-node processes. In this framework, a module or hyperedge failure occurs when the fraction of failed nodes in the hyperedge exceeds a threshold, and the failure of one hyperedge results in the failure of the remaining nodes within the hyperedge. We believe that the introduction of these mechanisms in hypergraphs will bring new understanding to the mechanisms of cascading failure on complex systems and the conditions for global failure.

Percolation theory provides a unified framework for investigating the robustness and vulnerability of complex systems when some nodes suffer attack or random losses initially [28,29]. Specifically, ordinary bond(site) percolation is used to study the static connectivity of a network after removing some nodes (edges) [30,31]. It has been found that a network collapses as a continuous phase transition for networks with scale-free or Poisson degree distribution, i.e., the size of the giant component decreases to zero continuously when the initial damaged nodes or links exceeds a critical threshold [32,33]. When the node-to-node interaction is introduced to the percolation process, an iterative process of nodal removal can be triggered by initial failures, and a network disintegrates abruptly as a first-order phase transition [34], such as k-core percolation [35], bootstrap percolation [36] and articulation point percolation [37]. Similarly, the percolation dynamics on interdependent or multilayer networks are also iterative processes and exhibit a first-order phase transition where the mutually connected giant component collapses abruptly at the percolation threshold [38–41]. Recently, the critical properties of percolation on networks with higher-order interactions are attracting more and more attention, such as percolation on hypergraphs with uniform and Poisson cardinality distribution [42], kagome hypergraph [43], multiplex hypergraphs [44], higher-order dependent networks [45], and core percolation on hypergraphs [46]. Percolation theory has been successfully applied to various types of networks and cascading failure models. The extension of percolation theory to hypergraphs with higher-order interactions, combined with the threshold rule, can provide new insights into the mechanisms of cascading failure on complex systems, and thus provides a unified framework for investigating the behavior of complex systems with higher-order interactions.

In this paper, we explore a threshold model of cascading failure on hypergraphs using percolation theory. Our model assumes that if the fraction of failed nodes in a hyperedge exceeds a threshold ϕ, the entire group and its residual members will be removed. Our main finding is that the robustness of hypergraphs exhibits a non-monotonic dependence on the average hyperdegree of nodes, with both high and low values leading to lower robustness. We demonstrate this effect by varying the average hyperdegree of nodes in two ways: (1) fixing the number of hyperedges by tuning the average cardinality, and (2) fixing the average cardinality by adjusting the number of hyperedges. Furthermore, the threshold parameter ϕ plays a significant role in determining the percolation transition types in the cascading dynamics, with a lower value of ϕ resulting in a discontinuous collapse and a larger value leading to a continuous disintegration. Our results suggest that controlling the average hyperdegree of nodes to a moderate level could optimize the robustness of a network, and enhancing the redundancy design of a hypergraph could reduce the sensitivity of nodes and improve the resilience of the hypergraph.

2. Model

We construct a hypergraph $H(N, M)$ which is composed of N nodes and M hyperedges. The number k of hyperedges attached to a node is defined as its hyperdegree, and it follows the hyperdegree distribution $P(k)$. Additionally, a hyperedge may have m nodes, which is named as its cardinality. We assume cardinality m conforms to the distribution $Q(m)$. Each hyperedge on average contributes $\langle m\rangle \equiv \sum Q(m)m$ to the total hyperdegree of the hypergraph, and the average hyperdegree is $\langle k\rangle = M\langle m\rangle/N$. From a practical viewpoint of network topology, a hyperedge can be seen as a functional module. We assume that when the fraction of failed nodes in the hyperedge reaches a threshold ϕ, the whole module will fail. With this dynamic mechanism, a fraction p of the initially removed nodes will trigger a cascading failure by the iteration of two processes: (1) the first one is the *node-to-edge* failure, i.e., the malfunctions of some nodes could lead to the failure of the whole hyperedge as well if the proportion of failed nodes in it exceeds the threshold ϕ; (2) the second one is the *edge-to-node* failure, i.e., when a hyperedge fails, all the remaining nodes attached to this hyperedge will also fail. When a cascading failure process is triggered, some hyperedges will fail first as the initially failed nodes can be distributed in different hyperedges, which is the node-to-edge failure process. Once a hyperedge fails, the remaining nodes in the hyperedge will malfunction simultaneously, which is the edge-to-node failure process. After some iterative steps of these two processes, the system will reach a stable state with no more hyperedge or node failure (See Fig. 1 for an illustration).

In our model, we use the threshold parameter ϕ to quantify the tolerance of a hyperedge to its failed members. When $\phi \to 1$, nodes within the same hyperedge become independent, and failures cannot spread from one hyperedge (node) to another. In contrast, $\phi \to 0$ implies that the nodes within the same hyperedge become completely dependent. In other words, if one node in the hyperedge fails, all nodes in the hyperedge will definitely fail. The parameter ϕ determines the severity of the cascading failure in the system.

To evaluate the structural integrity of the system, we use the final relative size $S \equiv G/N$ of the giant component in the stable state, as in previous works [7], where G is the number of nodes in the giant component. A network with $S = 0$ is considered to be completely destroyed. To study the robustness of the hypergraph after removing a fraction $1 - p$ of nodes, we focus on the threshold p_c, above which the final network can have a state of $S > 0$. Generally, a small critical point p_c suggests a more robust network, as it signifies that the giant component can exist after the cascading failure process, even in the case of initially removing a more significant number of nodes.

3. Theoretical and simulation results

3.1. The final survival probability of a random node in the hypergraph

Considering a random node with hyperdegree k, it can survive at the end of cascading failure only if all of its incident hyperedges exist. Assuming that T is the final survival probability of a random hyperedge reached by a random node, the survival probability $\hat{T}$ of the random node after the cascading process can be given by

$$\hat{T} = p\sum_k P(k)T^k. \tag{1}$$

Since one hyperedge can package m different nodes, which obey the distribution $Q(m)$, it can be inferred that a randomly chosen node belongs to a hyperedge of cardinality m with probability $Q(m)m/\langle m\rangle$.

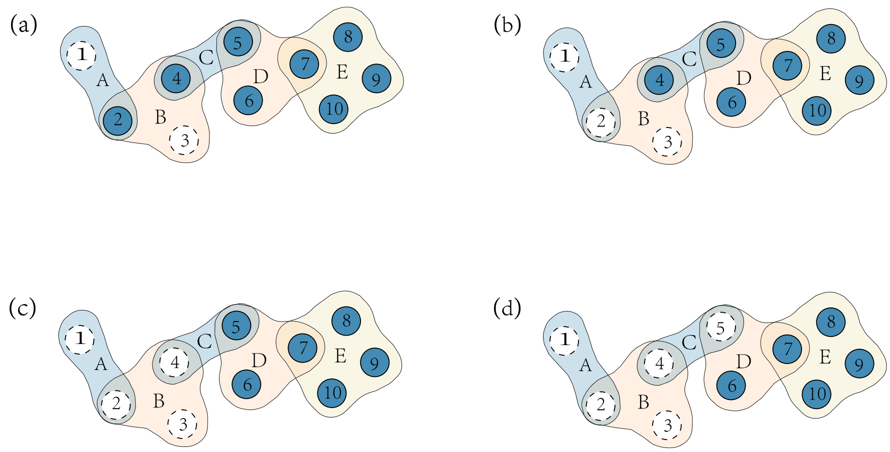

Fig. 1. Schematic diagram of the process of cascading failure on a hypergraph composed by $N = 10$ nodes and $M = 5$ hyperedges with the threshold $\phi = 0.4$, where the enclosed nodes form a hyperedge. (a) Two nodes 1 and 3 in the hypergraph was removed initially; (b) Since the fraction of failure nodes in hyperedge A exceeds the threshold ϕ, the hyperedge goes to fail and the node 2 fails as well; (c) then the fraction of total failed nodes in hyperedge B exceeds the threshold ϕ, and the hyperedge B and node 4 also fails; (d) the hyperedge C will fail since the fraction of failed nodes exceeds the threshold ϕ, and the system reaches a stable state in the end.

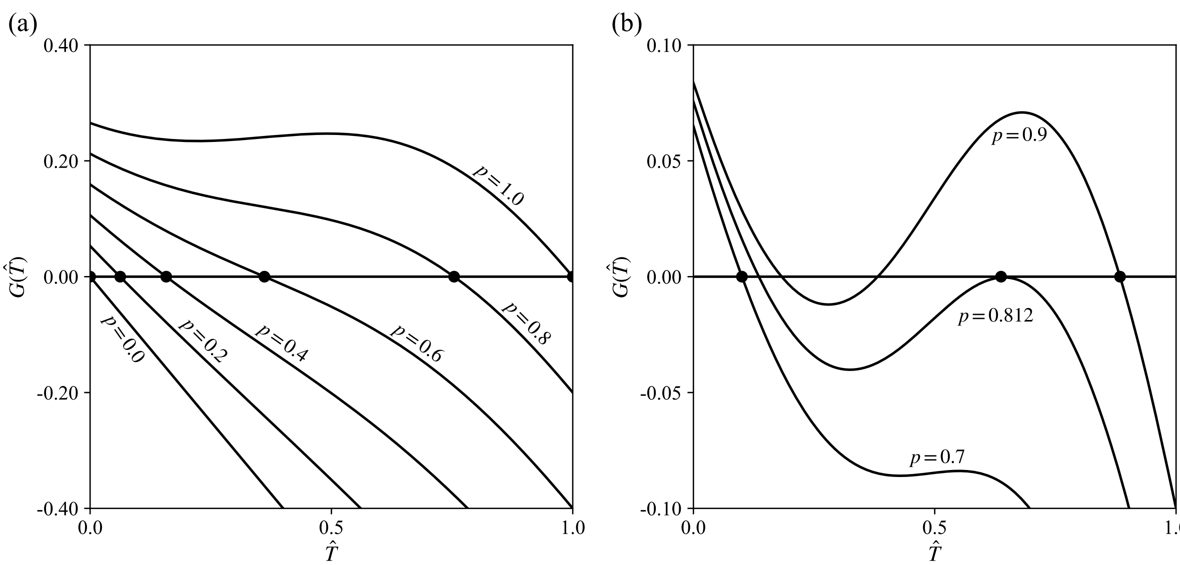

Fig. 2. The function $G(\hat{T})$ for random hypergraphs with different values of p, where the black dots on the horizontal axis are the solutions of $G(\hat{T}) = 0$. (a) the results for $\langle m\rangle = 1.5$, $\langle k\rangle = 3$ and $\phi = 0.5$; (b) the results for $\langle m\rangle = 2$, $\langle k\rangle = 4$ and $\phi = 0.5$.

Therefore, in terms of the probability $\hat{T}$, the probability T can be written as the following form

$$T = \sum_{m=0}^{\infty} \frac{Q(m)m}{\langle m\rangle} \sum_{n=0}^{m-1} \binom{m-1}{n} \hat{T}^{m-n-1}(1-\hat{T})^n F(m,n), \tag{2}$$

where n is the number of failed nodes and follows a binomial distribution. $F(m,n)$ is the response function with $F(m,n) = 1$ if $n \le m\phi$ and $F(m,n) = 0$ otherwise.

By inserting Eq. (2) into Eq. (1), we can get $\hat{T}$ by solving the following equation

$$G(\hat{T}) = 0 \tag{3}$$

with the function $G(\hat{T})$ defined as

$$G(\hat{T}) = p \sum_k P(k) \Big\{ \sum_{m=0}^{\infty} \frac{Q(m)m}{\langle m\rangle} \sum_{n=0}^{m-1} \binom{m-1}{n} \hat{T}^{m-n-1}(1-\hat{T})^n F(m,n) \Big\}^k - \hat{T}. \tag{4}$$

In this paper, we consider a simple case where both the cardinality m and the hyperdegree k follow the Poisson distribution with average $\langle m\rangle$ and $\langle k\rangle$, respectively. From Fig. 2(a), we can find that the final survival probability $\hat{T}$ of a random node changes from 0 to 1 continuously in the variation of the parameter p from 0 to 1 for $\langle m\rangle = 1.5$, $\langle k\rangle = 3$ and $\alpha = 0.5$. In this case, the poor overall connectivity of the hypergraph limits the propagation of the initial failures. Therefore, only by removing all the nodes can eradicate the whole hypergraph. However, for Fig. 2(b), we can find that the final fraction $\hat{T}$ of survival nodes can change abruptly from $\hat{T}_{c2}$ to $\hat{T}_{c1}$ at a critical point $\hat{p}_c \approx 0.812$ with the increase of p for $\langle m\rangle = 2$, $\langle k\rangle = 4$ and $\alpha = 0.5$. When the function curve $G(\hat{T})$ is tangent with the horizontal axis, the discontinuous jumps occur, and the jump point $\hat{p}_c$ can be obtained by solving the Eq. (5) and Eq. (3) simultaneously

$$\frac{dG(\hat{T})}{d\hat{T}}\Big|_{\hat{T}_{c1}} = 0. \tag{5}$$

In general, the value $\hat{T}_{c2}$ that the fraction of final survival nodes jumps to is not 0 but a small value, which is because there are always some nodes distributed in some small fragments. But as $\langle m\rangle$ gets larger, $\hat{T}_{c2}$ gets closer and closer to 0 (see in Fig. 3). Fig. 3 displays the simulation and theoretical results of the final fraction $\hat{T}$ of survival nodes versus p, from which one can find the theoretical results are

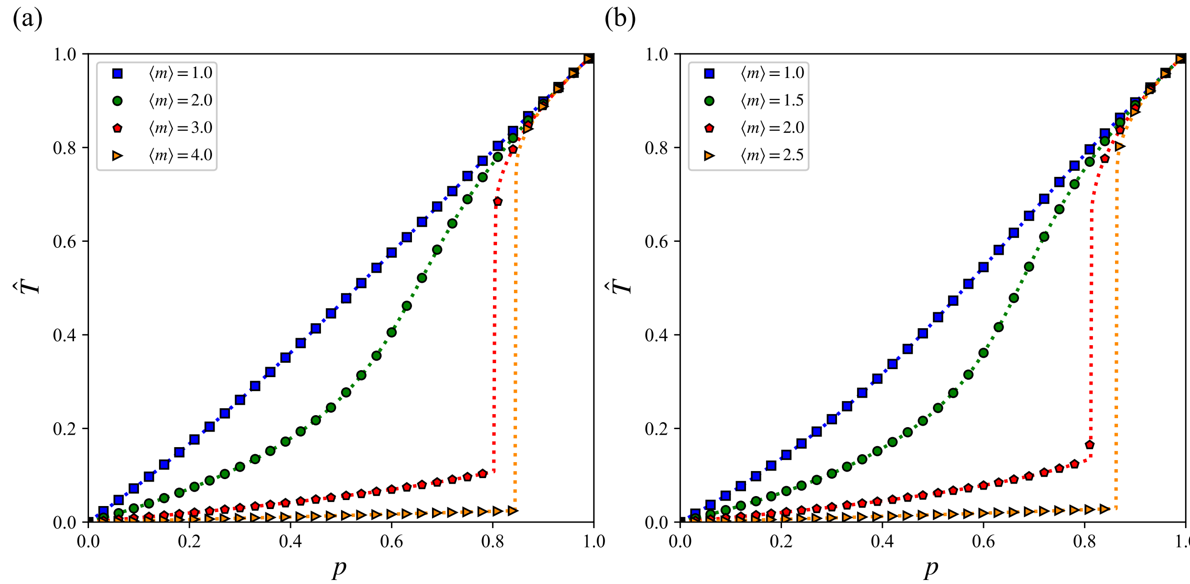

Fig. 3. The final fraction $\hat{T}$ of survival nodes versus p for different average cardinality $\langle m \rangle$ with $\phi = 0.5$, where the symbols represent simulation results, and dashed lines represent theoretical predictions. (a) The results for the number of hyperedges $M = N$; (b) the results for the number of hyperedges $M = 2N$. In both panels, the hypergraph size is $N = 10^5$.

in good agreement with the simulation. Thus the correctness of our theory can be verified. At the same time, one can also find that, for small average cardinalities $\langle m \rangle$, the final fraction $\hat{T}$ of surviving nodes increases continuously with the variation of p from 0 to 1, and for large average cardinalities $\langle m \rangle$, $\hat{T}$ first increases slowly and then increases suddenly to a considerable value at a jump point $\hat{p}_c$, which validates the results of Fig. 2. All these results prove that a larger $\langle m \rangle$ usually makes the initially failed nodes spread to a wider range and thus leads to more failed nodes in the steady state.

3.2. The size of the giant component and the percolation transition

With the final survival probability $\hat{T}$ of a random node, we aim to solve the probability S of a random node existing in the final giant component. Firstly, we solve the final cardinality distribution $Q'(m)$ by the survival probability $\hat{T}$ of the random node after the cascading process

$$Q'(m) = \sum_{m' \geq m} Q(m') \binom{m'}{m} \hat{T}^m (1 - \hat{T})^{m'-m} F(m, m' - m). \tag{6}$$

Based on $Q'(m)$, we can get the average cardinality $\langle m \rangle' = \sum_m Q'(m)m$ of hyperedges in the finally steady stage. Since the nodes in a hyperedge is picked randomly, we can infer that the final hyperdegree distribution $P'(k)$ of nodes follows the Poisson distribution with an average $\langle k \rangle' = \langle m \rangle' / \langle m \rangle \langle k \rangle / \hat{T}$. After that, we solve the probability R of a random hyperedge reached by a random node being connected to the giant component and the probability $\hat{R}$ of a random node in a random hyperedge being connected to the giant component. For a random hyperedge of cardinality m reached by a random node, the probability that at least one of the remaining $m - 1$ nodes can be connected to the giant component is $1 - (1 - \hat{R})^{m-1}$, and the probability R is

$$R = \sum_m \frac{Q'(m)m}{\langle m \rangle'} [1 - (1 - \hat{R})^{m-1}]. \tag{7}$$

Analogously, the probability $\hat{R}$ is

$$\hat{R} = \sum_k \frac{P'(k)k}{\langle k \rangle'} [1 - (1 - R)^{k-1}]. \tag{8}$$

With R and $\hat{T}$, we can get the probability S that a random node in the giant component, i.e., the fraction of nodes in the giant component, which is

$$S = \hat{T} \sum_k P'(k)[1 - (1 - R)^k]. \tag{9}$$

By imposing that the largest eigenvalue of the Jacobian matrix of Eqs. (7) and (8) is equal to one at $R = \hat{R} = 0$, we can find the condition for the second-order percolation transition

$$\frac{\langle k(k-1) \rangle'}{\langle k \rangle'} \frac{\langle m(m-1) \rangle'}{\langle m \rangle'} = 1, \tag{10}$$

where the angle brackets $\langle \cdot \rangle'$ denote the average of a given quantity computed over the final cardinality distribution $Q'(m)$ or final hyperdegree distribution $P'(k)$.

Since the nodes in the giant component are part of the surviving nodes, the drastic change of $\hat{T}$ will also lead to a sudden variation in S at the changing point $\hat{p}_c$. For lower values of average cardinality $\langle m \rangle$, the system undergoes a single second-order percolation transition without any abrupt changes. However, as $\langle m \rangle$ increases, the system exhibits an abrupt change in the fraction of surviving nodes $\hat{T}$, leading to a sudden variation in the giant component S at the transition point $\hat{p}_c$. This can result in two scenarios: a double phase transition involving both second-order and first-order transitions when $\hat{T}_{c2}$ is greater than the critical value $\hat{T}_c$ for the emergence of the giant component, or a single first-order percolation transition when $\hat{T}_{c2}$ decreases to $\hat{T}_c$ or below. Thus, the system can undergo a first-order phase transition, a second-order phase transition, or a double phase transition depending on the values of $\langle m \rangle$, $\langle k \rangle$, and ϕ.

Fig. 4 displays the emergence of the giant components of hypergraphs with the increase of p for Poisson cardinality distribution and Poisson hyperdegree distribution. Firstly, we observe that the emerging ways of the giant components are very different for different average cardinality $\langle m \rangle$. When $\langle m \rangle$ is small, the giant component emerges continuously as p exceeds a percolation transition point p_c^{II}, e.g., $\langle m \rangle = 2$ in Fig. 4(a) or $\langle m \rangle = 1.5$ (b). However, for a large $\langle m \rangle$, the giant component first emerges continuously when p reaches the percolation transition point p_c^{II}, and then, the giant component will exhibit another sudden increase at a first-order phase transition point p_c^I, which is a double phase transition and corresponds to the case of $\langle m \rangle = 2.5$ for $M = 2N$ or $\langle m \rangle = 2$ for $M = 4N$ shown in Fig. 4(a) or (b), respectively. As $\langle m \rangle$ is further increased, the percolation of the system transforms into a single first-order phase transition, and the size of the giant component emerges discontinuously at the first-order percolation transition point

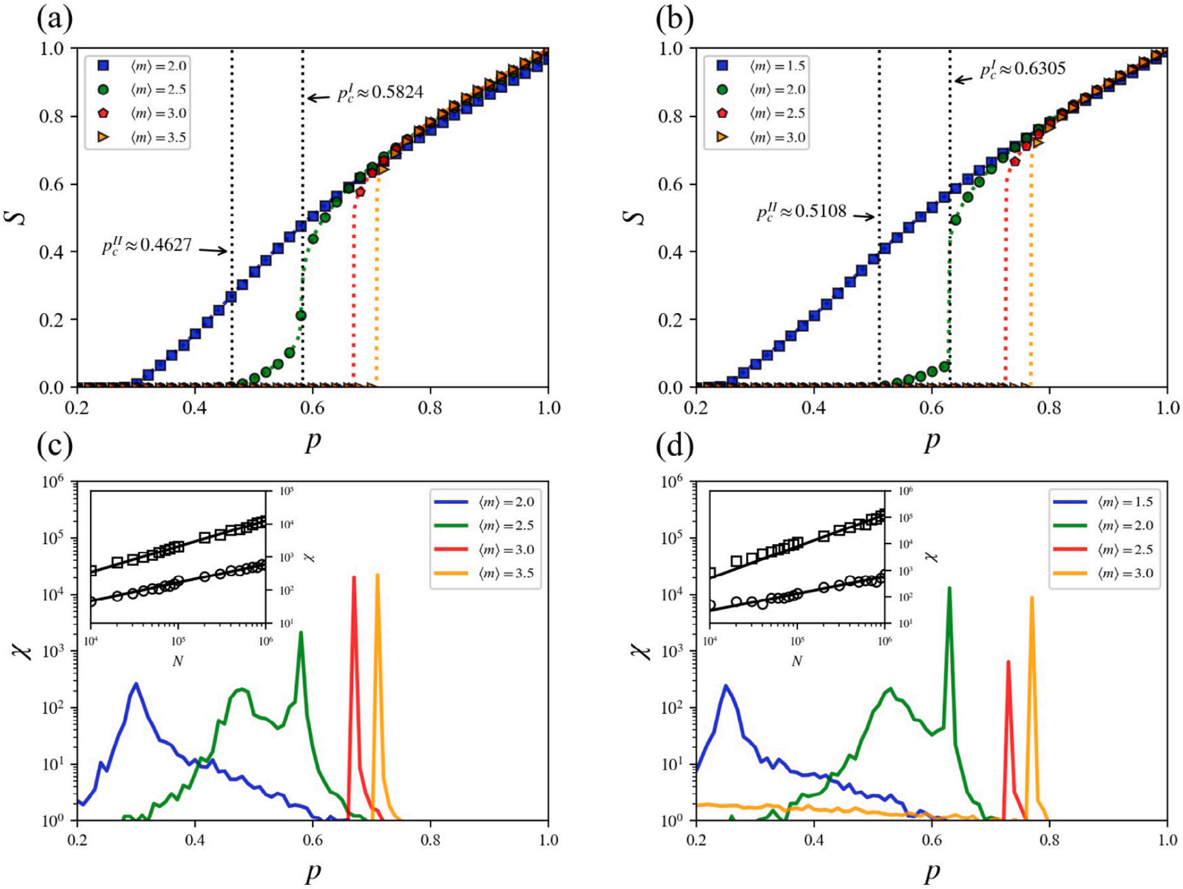

Fig. 4. The percolation transition for different average cardinality $\langle m \rangle$ with $\phi = 0.7$. (a–b) The final relative size S of the giant component as a function of the node reserving probability p for the number of hyperedges $M = 2N$ and $M = 4N$, respectively. The symbols represent simulation results for the system size $N = 10^5$, and dashed lines represent theoretical predictions; (c–d) The simulation results for susceptibility χ versus p under the same parameter settings in (a–b) respectively. The insets of (c–d) show the asymptotic divergence of susceptibility χ at the first-order and second-order phase transition points with the increase of the system size N in a double logarithmic scale when the system undergoes a double phase transition, i.e., $\langle m \rangle = 2.5$ for $M = 2N$ and $\langle m \rangle = 2$ for $M = 4N$, respectively, where the squares and circles represent simulation results for the first-order and second-order phase transition respectively, and the solid lines denote the fitting results.

p_c^I, e.g., $\langle m \rangle = 3$ in Fig. 4(a) or $\langle m \rangle = 2.5$ (b). Referring to Fig. 4(c), the size G of the giant component exhibits the maximal fluctuations at a phase transition point as the susceptibility $\chi = [\langle G^2 \rangle - \langle G \rangle^2]/\langle G \rangle$ diverges at the critical point as N tends to infinity. Hence, we can determine the percolation transition point via numerical simulations by analyzing χ for a large N. Furthermore, the double-peak structure of χ indicates that the system undergoes a double phase transition for the specific value of $\langle m \rangle$. The inset of Fig. 4(c) shows the susceptibility χ as functions of the system size N at the first-order phase transition point p_c^I and the second-order percolation transition point p_c^{II} for the case of double phase transition, i.e., $\langle m \rangle = 2.5$, $M = 2N$ and $\phi = 0.7$, which confirms the divergence of χ at the percolation transition points when the system size $N \to +\infty$ and the presence of double phase transition. Similarly, Fig. 4(d) also validates the presence of double phase transition for the parameter settings $\langle m \rangle = 2$, $M = 4N$ and $\phi = 0.7$.

The average cardinality $\langle m \rangle$ plays a crucial role in determining the form of percolation transitions and the robustness of the hypergraph. To explore the value of $\langle m \rangle$ that can optimize the robustness of the hypergraph and leads to the lowest percolation point for a given combination of parameters $\langle k \rangle$ and ϕ, we plot the percolation transition point p_c^I or p_c^{II} versus $\langle m \rangle$ for a fixed number M of hyperedges with $\phi = 0.5$ and $\phi = 0.7$ in Fig. 5(a) and (b), respectively. We observe that the percolation transition point p_c^I or p_c^{II} initially decreases, and then increases with the increase of $\langle m \rangle$. For a small $\langle m \rangle$ (region III), the hypergraph is in a state of fragmentation, and the giant component of the hypergraph does not exist even if no node is removed from the hypergraph. For a larger $\langle m \rangle$ (region II), the giant component of the hypergraph emerges continuously when $p > p_c$. As $\langle m \rangle$ increases further (region (II, I)), the system exhibits a double phase transition, where the system percolates as a second-order phase transition firstly and then undergoes a first-order phase transition. In the last region I, the second-order percolation transition disappears, and the system transforms into a single first-order percolation transition. Furthermore, we plot the phase transition point p_c^I or p_c^{II} versus the average hyperdegree $\langle k \rangle$ for a fixed average cardinality $\langle m \rangle = 2$ with $\phi = 0.5$ and $\phi = 0.7$ in Fig. 5(c) and (d), respectively. We find that there is also a non-monotonic effect of the percolation point p_c versus the average hyperdegree $\langle k \rangle$, and four distinct regions: fragmentation region III, second-order phase transition region II, first-order phase transition region I, and double phase transition region (I, II). A lower $\langle k \rangle$ or $\langle m \rangle$ tends to correspond to poorer connectivity, limiting the failure propagation in the network and hindering the formation of the giant component. However, a large $\langle m \rangle$ or $\langle k \rangle$ leads to a well-connected hypergraph, making it more vulnerable to failures. Therefore, a moderate $\langle k \rangle$ or $\langle m \rangle$ enables the hypergraph to achieve optimal robustness, balancing the competing mechanisms of failure propagation and connectivity.

4. Conclusion

Higher-order interactions are ubiquitous in complex systems, and their impact on the robustness of such systems is of great importance. In this paper, we aimed to explore the robustness of complex systems with higher-order interactions by investigating cascading failures

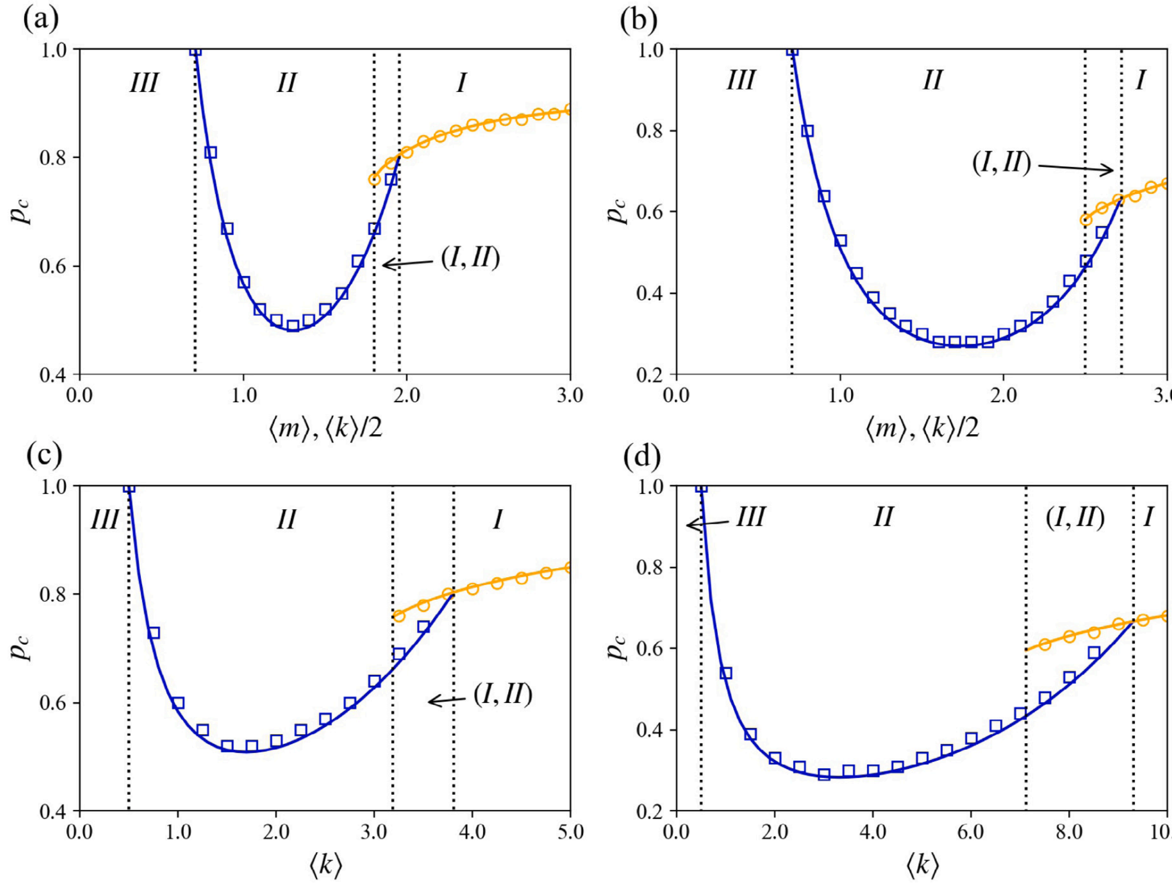

Fig. 5. The percolation transition point $p_c(p_c^I$ or $p_c^{II})$ as functions of the average cardinality $\langle m \rangle$ or average hyperdegree $\langle k \rangle$. (a–b) shows the percolation transition point p_c^I or p_c^{II} versus the average cardinality $\langle m \rangle$ with the number of hyperedges $M = 2N$ for $\phi = 0.5$ and $\phi = 0.7$, respectively; (c–d) shows the percolation transition point p_c^I or p_c^{II} versus the average hyperdegree $\langle k \rangle$ with $\langle m \rangle = 2$ for $\phi = 0.5$ and $\phi = 0.7$, respectively. The symbols represent simulation results obtained by locating the peak of the susceptibility χ, which is calculated from 100 independent realizations of the system with $N = 10^5$ for each value of p. The blue solid lines denote the second-order percolation points predicted by Eq. (10), and the orange solid lines denote the first-order phase transition points predicted by Eqs. (5) and (3). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

on hypergraphs. To model higher-order interactions in the cascading process, we employed a threshold rule to account for both the node-to-edge and edge-to-node processes, whereby the failure of some nodes in one hyperedge triggers the malfunction of the entire hyperedge, and the malfunction of one hyperedge leads to the failures of the remaining nodes in the hyperedge. This novel interaction mechanism is absent in pairwise interactions, where the failure of one node causes the failure of the entire edge. Conversely, for higher-order interactions involving multiple nodes, the failure of some individual nodes may not necessarily lead to the failure of the entire group.

Our findings reveal that the cascading process can be triggered by the removal of a fraction of randomly selected nodes, and the failures of nodes and hyperedges occur alternately in the system, resulting in a significant reduction in the fraction of final survival nodes and the sizes of the giant components. Our model describes several essential properties of realistic scenarios, including the cumulative effect in cascading failures, the sensitivity of a node to external disturbances or the strength of nodal interdependence of a group, and the feature of the redundant design of functional modules or emergency management plans in complex systems.

Moreover, we have also find that hyperedges in a hypergraph can have both positive and negative impacts on the network's robustness. On the one hand, hyperedges can strengthen the connections between nodes and promote the emergence of the giant component. On the other hand, hyperedges can also increase the risk of failure propagation in the network, potentially leading to its collapse if it is attacked. Although the duality of edges exists in the threshold model of simple networks with pairwise interactions [1], our findings confirm the generality of this phenomenon in the cascading of complex systems. At the same time, this duality also highlights the importance of carefully considering the impact of hyperedges on a hypergraph's robustness and not simply assuming that adding more connections will always improve its resilience.

Our model also demonstrates that the hypergraph may collapse either in a discontinuous or continuous manner, corresponding to a first- or second-order percolation transition, respectively. Additionally, the size of the giant component can suddenly increase at another first-order phase transition point after the percolation of a hypergraph as a second-order phase transition manner. The type of percolation transitions is dependent on the failure threshold of hyperedge, average hyperdegree, and average cardinality. These findings can aid in controlling the ways of collapse by adjusting the average hyperdegree or average cardinality of networks.

In summary, we have introduced the threshold rule to the cascading models on hypergraphs to depict the node-to-edge failure mechanism. Base on this model, we have also extended the theoretical framework of percolation to the investigation of cascading failures in complex systems with higher-order interactions. Our research shares similarities with the work by Xu et al. [27] in the exploration of cascading on hypergraphs by the threshold rule and the identification of dual effects of hyperedges. However, there are some differences. We considered both node-to-edge and edge-to-node interactions, whereas the work by Xu et al. only focused on node activation when the proportion of activated neighbors exceeded a threshold. Secondly, Xu et al. focused on the conditions for global failure occurrence triggered by the failure of one random node [27], while our study examined the influence of the initial fraction of failed nodes. Additionally, we reported the presence of double-percolation transitions as the initial fraction of failed nodes changes. Our study contributes to understanding the dynamics of networks with higher-order interactions and their resilience to cascading failures. We hope that our findings, together with other works on hypergraphs [27,44–47] and more, will contribute to the ongoing research in this field and inspire further investigations into the topic. However, we recognize that there is still much to learn about the behavior of complex systems, and our model is just one step towards a more complete understanding of this challenging problem.

CRediT authorship contribution statement

Run-Ran Liu: Conceptualization, Investigation, Methodology, Validation, Supervision, Writing – original draft. **Chun-Xiao Jia:** Conceptualization, Supervision, Software, Validation. **Ming Li:** Software, Validation, Writing – review & editing. **Fanyuan Meng:** Software, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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